This is an electronic version of an article whose final and definitive form has been published in Scripta Materialia (Volume 57, 2007, Pages 177–178) (Elsevier); Scripta Materialia is available online at: http://www.sciencedirect.com/.

# Comment on "A growth rule for regular polygons with arcs as sides"

by A.Morawiec

Laboratoire d'Etude des Textures et Application aux Matériaux,
Université Paul Verlaine – Metz,
Ile du Saulcy, F-57045 Metz, France
Tel.: (+33)387315325, Fax:(+33)387315377
E-mail: nmmorawi@cyf-kr.edu.pl

**Abstract**

In a recent publication [*Scripta Mater.* **56** (2007) 37], the rate of curvature driven growth of two dimensional regular cells was calculated. It was claimed that the formula for the rate differs from the von Neumann's law. However, the calculations leading to that conclusion were inconsistent with the von Neumann's kinetic postulate underlying the law. We show that after correcting the error, the model of a regular cell leads to the von Neumann's relationship.

*Keywords*: Coarsening; Grain growth; von Neumann's law

In a recent article, Nordbakke [1] considered curvature driven growth of two dimensional regular cells. The author derived a formula for the rate of their area change as a function of the number of cell sides. That expression considerably differs from the well–known von Neumann's law [2]. However, the paper does not explain the reasons for the discrepancy. Below, we sort out what was actually calculated in [1]. Then the derivation leading to the von Neumann's law is given. We use the notation of [1] (plus two additional symbols, $f_n$ and $\alpha_0$, defined below). We also refer to points $O, A, B, C$, which are marked in Fig. 2 of [1]. The model is a regular $n$–sided cell with circular arcs of radius $R$ as boundaries. The boundaries move towards the centers of the circles with velocities proportional to their curvature. With internal vertex angles of the cell equal to $2\pi/3$, the model has geometric features of a foam bubble.

In the calculations of [1], a tacit assumption is that the angle supporting a boundary arc ($2\alpha$) is constant. With constant $\alpha$ and changing length of the chord $AB$, the center ($C$) of the boundary arc must move with respect to the center of the cell ($O$). In this case, due to the shrinkage of $R$, the cell area gets smaller even if $n$ is larger than 6. To eliminate this obvious difficulty, the additional factor sign$(n-6)$ was 'artificially' introduced in Eqs (6) and (7) of [1].

Now, in the von Neumann's kinetic postulate of proportionality between the boundary velocity and its curvature, the velocity is presumed to be given with respect to an immobile external reference frame. When writing the postulate in the form of Eq.(5), one accepts that the centers of the boundary arcs (point $C$) are fixed in that frame. In other words, their distance to the cell center $O$ is constant. Hence, the angle $\alpha$ depends on $R$. This means that the expression for the derivative $\mathrm{d}A/\mathrm{d}R$ used in [1] and, consequently, the final formula given as Eq.(7) are not consistent with the von Neumann's kinetic postulate . Needless to say, the discussion which follows Eq.(7) is groundless[1].

The correct calculation is slightly more involved than that of [1]. The expression (4) of [1] for the cell area $A$ can be written as

$$A = A(R) \equiv R^2 f_n(\alpha) \ ,$$

where $f_n(\alpha) = n\left(2\alpha + 2\cot(\pi/n)\sin^2(\alpha) - \sin(2\alpha)\right)/2$, and $\alpha$ depends on $R$. Due to the kinetic postulate ($\mathrm{d}R/\mathrm{d}t = -C/R$, $C = \mathrm{const} > 0$),

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -C\left(2f_n(\alpha) + R\frac{\mathrm{d}f_n}{\mathrm{d}\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}R}\right) \ .$$

With the internal vertex angles equal to $2\pi/3$, one has $\alpha = \alpha_0 \equiv \pi\left(1/n - 1/6\right)$. The values of $f_n$

[1] Moreover, there are misprints in expressions for $t_2$ and $t_1 \cdot t_2$.

and $\mathrm{d}f_n/\mathrm{d}\alpha$ at $\alpha = \alpha_0$ are

$$f_n(\alpha_0) = \pi + \frac{n}{12}\left(3\cot\left(\frac{\pi}{n}\right) - 3\sqrt{3} - 2\pi\right) \quad \text{and} \quad \frac{\mathrm{d}f_n}{\mathrm{d}\alpha}(\alpha_0) = \frac{n}{2}\left(3 - \sqrt{3}\cot\left(\frac{\pi}{n}\right)\right) ,$$

respectively. The calculation of $R\,(\mathrm{d}\alpha/\mathrm{d}R)$ is more tedious but still elementary. Here is a sketch of it for the case $n < 6$ shown in Fig. 2. From the law of cosines for the triangle $OCB$

$$R^2 = |OC|^2 + r^2 - 2\,|OC|\; r\,\cos(\pi - \delta/2) ,$$

where $r = R\sin(\alpha)/\sin(\delta/2)$. One can verify by substitution that one of the solutions of this equation is $R = |OC|\,\csc(\delta/2 - \alpha)\sin(\delta/2)$. Hence, $R\,(\mathrm{d}R/\mathrm{d}\alpha)^{-1} = \tan(\delta/2 - \alpha)$. With $\delta = 2\pi/n$ and $\alpha = \alpha_0$, one gets

$$\left(R\,\frac{\mathrm{d}\alpha}{\mathrm{d}R}\right)_{|\alpha=\alpha_0} = \frac{\sqrt{3}}{3} .$$

Analogous calculation for $n > 6$ gives the same value. Substitution into the expression for the rate of area change gives $\mathrm{d}A/\mathrm{d}t = C\pi(n-6)/3$, i.e., Eq.(1) of [1]. In conclusion, the model of a regular cell bounded by circular arcs moving with constant velocities $-C/R$ with respect to fixed arc centers leads exactly to the von Neumann's relationship.

For completeness, the following remark is in place. The growth of a regular two dimensional cell with (three) circular arc boundaries was considered earlier by Gusak and Tu [3]. That analysis also leads to a formula different from the von Neumann's law. The discrepancies have an origin in different interpretation of the kinetic postulate. Circular arcs moving with constant velocities with respect to fixed arc centers violate the requirement that internal vertex angles must equal $2\pi/3$. Above, to get the von Neumann's formula, the rate $dA/dt$ is calculated at the instant when the angles reached this particular value. In the derivation of Gusak and Tu [3], as in [1], the vertex angles are fixed at $2\pi/3$, and – to maintain circular shape of boundaries – the arc centers are mobile. Differently than in [1], the boundary velocity of [3] is adjusted so it is the same as in the von Neuamann's case in the vicinity of vertices. Since the boundary velocities differ, the resulting expressions for the rate are different. An extensive discussion about the validity of these and other formulations of the kinetic postulate is beyond the scope of this note. We only point out that the scenario of Gusak and Tu [3] seems to be more realistic than the one leading to the von Neumann's law, and the case considered in [1] does not reflect features of curvature driven coarsening.

Finally, let us note that an analysis similar to that of [3] and [1] can be performed for three dimensional regular cells [4]. This would be another derivation of the rate of cell volume change as a function of the number of faces.

## References


[1] M.W. Nordbakke, *Scripta Mater.* **56** (2007) 37.

[2] J. von Neumann, in *Metal Interfaces*, American Society for Metals, Cleveland, OH, 1952, p.108.

[3] A.M. Gusak and K.N. Tu, *Acta Mater.* **51** (2003) 3895.

[4] M.E. Glicksman, *Philos. Mag.* **85** (2005) 3.